A Double Category Theoretic Perspective on Partiality
Work in progress

Adjoint School 2026 Group 2
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Introduction

Theorem Cockett and Garner 2014

Restriction categories are categories enriched in $\mathbb{P}\mathrm{StabLat}^{\mathrm{op}}$.

Outline

Restriction categories

Categories enriched in double categories

Bases of enrichment

Correspondence of distributors

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Restriction categories

Many frameworks where the notion of a "partially defined" arrow needs to be studied, e.g. sets and partial functions. Ordinary categories cannot capture such notion; we need extra structure to capture "partiality".

Definition [Cockett and Lack 2002]

A **restriction category** is a category $\mathcal{C}$ equipped with a **restriction structure**, that is an operator $\mathcal{C}(c,d) \to \mathcal{C}(c,c) : f \mapsto \overline{f}$, for all $c, d \in \mathcal{C}_0$ so that

(R1) $f\overline{f} = f$ for any f,

(R2) $\overline{f}\,\overline{g} = \overline{g}\,\overline{f}$ for any f, g with $\mathrm{dom}(f) = \mathrm{dom}(g)$,

(R3) $\overline{f\overline{g}} = \overline{f}\,\overline{g}$ for any f, g with $\mathrm{dom}(f) = \mathrm{dom}(g)$, and

(R4) $\overline{h}f = f\overline{hf}$ for any f, h with $\mathrm{cod}(f) = \mathrm{dom}(h)$.

Intuitively, $\overline{f}$ is determining where exactly on $\mathrm{dom}(f)$ is f defined, measuring its "degree of partiality". In particular, f will be called **total** when $\overline{f} = 1_{\mathrm{dom}(f)}$.

Examples

1. Any category $\mathcal{C}$ can be trivially made into a restriction category if we define $\overline{f} = 1_c$ for any $f : c \to d$. All arrows are total.
2. The category Par of sets and partial functions where

$$\overline{f} = \begin{cases} x, & \text{if f(x) is defined} \\ \text{undefined}, & \text{otherwise} \end{cases}$$

   for any $f : c \to d$, is a restriction category. The total arrows are the ordinary total functions.
3. The opposite category $\mathbf{StabLat}^{\mathrm{op}}$ meet-semilattices (posets with binary meets and a top element) and stable maps (functions preserving the joins but not necessarily the top element) is a restriction category if we define for any $f : c \leftarrow d$, $\overline{f}$ to be

$$\overline{f}(x) = x \wedge f(\top).$$

   Total arrows are those stable maps that also respect the top element.

The fundamental (restriction) functor

For a restriction category $\mathcal{C}$, we can define a partial order on its hom-sets by taking

$$f \leq g \iff f = g\overline{f},$$

intuitively meaning that f and g agree whenever f is defined. This partial order is compatible with the composition making $\mathcal{C}$ into a 2-category.

Given two restriction categories $\mathcal{C}, \mathcal{D}$, a **restriction functor** $F : \mathcal{C} \to \mathcal{D}$ is a functor that respects the restriction structure, that is $F\overline{f} = \overline{Ff}$ for any arrow f. In other words, if $\mathcal{O}(c) := \{\overline{f} : c \to c | f \in \mathcal{C}\}$ then F induces a functor $\mathcal{O}(c) \to \mathcal{O}(Fc)$.

Moreover, $\mathcal{O}(c)$ can be made into a meet-semilattice by taking $\overline{f} \wedge \overline{g} = \overline{f} \circ \overline{g}$ and $\top = 1_c$ and for every arrow $c \to d$ we get a stable map $\mathcal{O}(d) \to \mathcal{O}(c)$. In particular, for every restriction category $\mathcal{C}$ we get a restriction functor

$$\mathcal{O} : \mathcal{C} \to \mathsf{StabLat}^{\mathsf{op}}$$

called the **fundamental functor**.

Restriction categories

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Review: double categories

A (weak) double category is a (pseudo-)category internal to Cat:

$$\mathbb{D}_0 \;\underset{\text{cod}}{\overset{\text{dom}}{\underset{\longleftarrow}{\overset{\longleftarrow}{-\ \text{id} \rightarrow}}}}\; \mathbb{D}_1 \;\xleftarrow{\ \odot\ }\; \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1$$

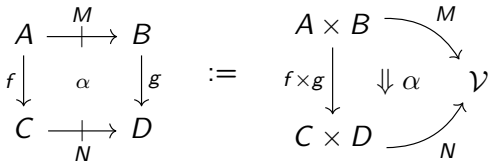
Concretely it consists of:

- objects A, B, ... (objects of $\mathbb{D}_0$);
- tight arrows $f : A \to B$ (morphisms of $\mathbb{D}_0$);
- loose arrows $f : A \nrightarrow B$ (objects of $\mathbb{D}_1$);
- cells $\begin{array}{ccc} A & \longrightarrow\!\!\!\!\!+ & B \\ \downarrow & \alpha & \downarrow \\ C & \longrightarrow\!\!\!\!\!+ & D \end{array}$ (morphisms of $\mathbb{D}_1$);

  + identity arrows, composition operations, (associator, unitors), axioms...

Example

Let $\mathcal{V}$ be a distributive monoidal category. The double category of matrices $\mathbb{M}\mathrm{at}(\mathcal{V})$ has sets and functions as objects and tight arrows, matrices $M : A \times B \to \mathcal{V}$ as loose arrows $M : A \nrightarrow B$, and cells given by



Composition of loose arrows is given by matrix multiplication:

$$(M \odot N)(a, c) := \bigsqcup_{b \in B} M(b, c) \otimes N(a, b)$$

If $\mathcal{V}$ is the poset $\{0 \leq 1\}$ (with tensor min and unit 1), then $\mathbb{M}\mathrm{at}(\mathcal{V}) = \mathbb{R}\mathrm{el}$ the double category of sets and relations.

Review: Enriched categories

Let $(\mathcal{V}, \otimes, I)$ be a monoidal category. A $\mathcal{V}$-category $\mathcal{C}$ is given by:

- a set $\mathrm{ob}(\mathcal{C})$ of objects;
- for every pair of objects A, B, a hom-object $\mathcal{C}(A, B) \in \mathcal{V}$;
- for every objects A, B, C, morphisms

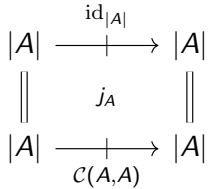
$$j_A : I \to \mathcal{C}(A, A) \qquad \text{and} \qquad \circ_{A,B,C} : \mathcal{C}(B, C) \otimes \mathcal{C}(A, B) \to \mathcal{C}(A, C)$$

subject to some axioms (composition is associative and unital...).

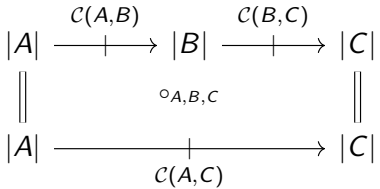
Enriching in a double category

Let $\mathbb{D}$ be a double category. A $\mathbb{D}$-category $\mathcal{C}$ is given by:

- a set $\mathrm{ob}(\mathcal{C})$ of objects;
- a function $|-| : \mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathbb{D})$;
- for every pair of objects A, B, a loose arrow $\mathcal{C}(A,B) : |A| \nrightarrow |B|$ of $\mathbb{D}$;
- for every objects A, B, C, cells



and



subject to some axioms (composition is associative and unital...).

Examples

- Let $\mathcal{V}$ be a monoidal category. We can see $\mathcal{V}$ as a one object and one tight arrow double category $\mathbb{V}$. A $\mathbb{V}$-category is exactly a $\mathcal{V}$-category.
  Conclusion: Categories enriched in double categories are direct generalizations of monoidal enriched categories.
- A $\mathbb{M}\mathrm{at}(\mathcal{V})$-category $\mathcal{C}$ is a collection $(X_A)_{A \in \mathrm{ob}(\mathcal{C})}$ of sets, and for each pair X_A, X_B a matrix

$$\mathcal{C}(A, B) : X_A \times X_B \to \mathcal{V} \qquad + \text{ conditions}$$

  Intuitively: $\mathcal{C}(A, B)$ can have many "inputs" and "outputs".

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Recovering restriction structure I

Reminder: Every restriction category $\mathcal{C}$ has its fundamental functor $\mathcal{O}_{\mathcal{C}} \colon \mathcal{C} \to \mathsf{StabLat}^{\mathsf{op}}$
Idea: Can we define restriction categories in terms of $\mathcal{O} = \mathcal{O}_{\mathcal{C}}$? I.e. do restriction categories correspond to functors into $\mathsf{StabLat}^{\mathsf{op}}$?
Not in general, but we know more about $\mathcal{O}$:

- $\mathcal{O}$ can be viewed as a 2-functor $\mathcal{C} \to \mathsf{StabLat}^{\mathsf{op}}$ (it preserves the ordering of morphisms)
- $\mathcal{O}$ is a local discrete fibration ($\mathcal{O} \colon \mathcal{C}(c, c') \to \mathsf{StabLat}^{\mathsf{op}}(\mathcal{O}(c), \mathcal{O}(c'))$ is a discrete fibration for every $c, c' \in \mathcal{C}$)

Recovering restriction structure II

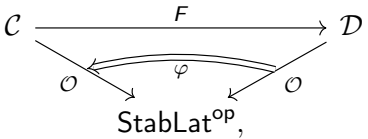
Theorem [Cockett and Garner 2014]

Every local discrete fibration $F\colon \mathcal{D} \to \mathsf{StabLat}^{\mathrm{op}}$ determines a unique restriction structure on $\mathcal{D}$ such that $\mathcal{O}_{\mathcal{D}} = F$.

What about restriction functors?

Theorem [Cockett and Garner 2014]

Restriction functors $F\colon \mathcal{C} \to \mathcal{D}$ correspond to pairs of a 2-functor $F\colon \mathcal{C} \to \mathcal{D}$ together with a 2-natural transformation φ



whose components are discrete fibrations.

Local discrete fibrations as enriched categories

Theorem [Cockett and Garner 2014]

For every 2-category $\mathcal{K}$, there is a double category $\mathbb{P}\mathcal{K}$ such that the 2-category of $\mathbb{P}\mathcal{K}$-categories is 2-equivalent to 2-Cat$/\!\!/_{ldf}\,\mathcal{K}$.

Corollary [Cockett and Garner 2014]

The 2-category of restriction categories is 2-equivalent to the 2-category of $\mathbb{P}(\mathsf{StabLat}^{\mathsf{op}})$-categories.

The construction $\mathbb{P}$ is coming out of nowhere, can we conceptualise it?

Proposition [our group]

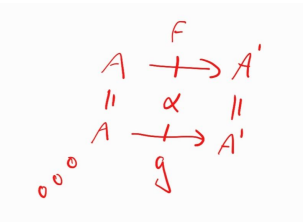
$\mathbb{P}\mathcal{K} = \widehat{\mathbb{D}\mathit{fib}\mathcal{K}}$

Local cocompletion of double categories

Definition [our group]

For every double category $\mathbb{D}$, the double category $\widehat{\mathbb{D}}$ has

- objects... objects of $\mathbb{D}$
- tight arrows... tight arrows of $\mathbb{D}$
- loose arrows $A \nrightarrow A'$... functors $\mathbb{D}(A, A')^{\mathrm{op}} \to \mathrm{Set}$
- cells... too technical



Conjecture

$\widehat{\mathbb{D}}$ is the *local cocompletion* of $\mathbb{D}$.

Lax slice of a double category

Theorem [Fujii and Lack 2024]

Given a bicategory $\mathcal{B}$ and a $\mathcal{B}$-category $\mathcal{C}$, there is a bicategory $\mathcal{B}/\mathcal{C}$ such that $(\mathcal{B}/\mathcal{C})\text{-Cat} \simeq \mathcal{B}\text{-Cat}/\mathcal{C}$.

Theorem [our group]

Given a double category $\mathbb{D}$ and a $\mathbb{D}$-category $\mathcal{C}$, there is a double category $\mathbb{D}/\!\!/\mathcal{C}$ such that $(\mathbb{D}/\!\!/\mathcal{C})\text{-Cat} \simeq \mathbb{D}\text{-Cat}/\!\!/\mathcal{C}$.

Proposition [our group]

The double category $\mathbb{P}\mathrm{StabLat}^{\mathrm{op}}$ is a sub-double category of $\Sigma\mathrm{Cat}/\!\!/\mathrm{StabLat}^{\mathrm{op}}$.

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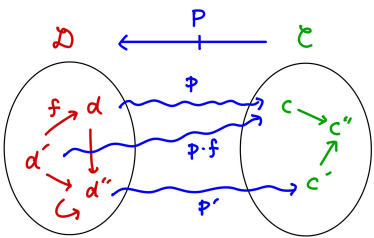
Correspondence of distributors

Distributors

Definition

A distributor $P\colon \mathcal{C} \nrightarrow \mathcal{D}$ (a.k.a. profunctor, bimodule) for categories $\mathcal{C}, \mathcal{D}$ is a functor $\mathcal{D}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Set}$.

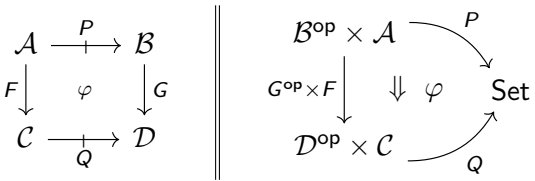
Intuitively, the set $P(d, c)$ is a collection of "morphisms" $p\colon d \rightsquigarrow c$.



sets :: functions :: relations $\approx$ *categories :: functors :: distributors*

Double categories of distributors

Categories, functors, distributors, and transformations form a double category $\mathbb{D}\mathrm{ist}$.



Similarly, for any double category $\mathbb{D}$, we get a virtual double category $\mathbb{D}$-$\mathbb{D}\mathrm{ist}$ of categories enriched in $\mathbb{D}$.
These virtual double categories can be thought of as stages for category theory, where categorical notions (weighted limits, cocompletion, etc.) can be defined and studied.

Restriction distributors

Definition [DeWolf 2017]

A restriction distributor $P\colon \mathcal{C} \nrightarrow \mathcal{D}$ for restriction categories $\mathcal{C}, \mathcal{D}$ is a distributor $\mathcal{C} \nrightarrow \mathcal{D}$ equipped with an operator $\overline{(-)}\colon P(d,c) \to \mathcal{O}(c)$ such that

(RD1) $p\overline{p} = p$ for any $p\colon d \rightsquigarrow c$,

(RD2) $\overline{p}\overline{f} = \overline{p}\overline{f}$ for any $p\colon d \rightsquigarrow c$ and $f\colon d \to d'$,

(RD3) $\overline{p}f = f\overline{pf}$ for any $f\colon d' \to d$ and $p\colon d \rightsquigarrow c$, and

(RD4) $\overline{g}p = p\overline{gp}$ for any $p\colon d \rightsquigarrow c$ and $g\colon c \to c'$.

$\overline{(-)}$ is the operator on "morphisms" $p\colon d \rightsquigarrow c$ that looks like the restriction operator of a restriction category.

Equivalence of the notion of distributors

Theorem [our group]

Restriction distributors $\mathcal{C} \nrightarrow \mathcal{D}$ correspond to distributors enriched in $\mathbb{P}\mathrm{StabLat}^{\mathrm{op}}$.

Conjecture

By defining the double category $\mathbb{RD}\mathrm{ist}$ of restriction categories, restriction functors, and restriction distributors, with a suitable notion of transformations, we would get

$$\mathbb{RD}\mathrm{ist} \quad \simeq \quad \mathbb{P}\mathrm{StabLat}^{\mathrm{op}}\text{-}\mathbb{D}\mathrm{ist}$$

This would improve the equivalence of the 2-categories of restriction categories and $\mathbb{P}\mathrm{StabLat}^{\mathrm{op}}$-enriched categories proved in [Cockett and Garner 2014].

Conclusion

Summary

- We found two new presentations of the double category, in which enriched categories are restriction categories.
- We extended this to an equivalence of distributors.

Future work

- Prove the conjectures.
- Study the two constructions of virtual double categories $\widehat{(-)}$ and $(-)/\!\!/(-)$ in more detail.

References I

- Cockett, R. and R. Garner (2014). "Restriction categories as enriched categories". In: *Theoretical Computer Science* 523, pp. 37–55.
- Cockett, R. and S. Lack (2002). "Restriction categories I: categories of partial maps". In: *Theoretical Computer Science* 270.1, pp. 223–259.
- DeWolf, D. (2017). "Restriction category perspectives of partial computation and geometry". PhD thesis. Dalhousie University.
- Fujii, S. and S. Lack (2024). "The oplax limit of an enriched category". In: *Theory Appl. Categ.* 40, No. 14, 390–412.

Thank you!